The Thermal-Convective Instability in a Stellar Atmosphere with Rotation and Hall Effects

R. C. Sharma

Department of Mathematics, Himachal Pradesh University, Simla-5, India

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The problem of thermal-convective instability of a stellar atmosphere is considered to include the effects due to rotation and Hall currents in the presence of a uniform vertical magnetic field. The criterion for monotonic instability is found to be unchanged by the presence of rotation and Hall effects.

1. Introduction

The instability in which motions are driven by buoyancy forces, of a thermally unstable atmosphere has been termed as "thermal-convective instability". Defouw [1] has generalized the Schwarzschild criterion for convection to include departures from adiabatic motion and has shown that a thermally unstable atmosphere is also convectively unstable, irrespective of the atmospheric temperature gradient.

Defouw has found that a stellar atmosphere is monotonically unstable if

$$D = \frac{1}{C_\mathrm{p}} (\mathcal{L}_\mathrm{T} - \varrho\, a\, \mathcal{L}_\varrho) + \varkappa\, k^2 < 0\,, \qquad (1)$$

where $\mathcal{L}$ is the energy lost minus the energy gained per gram per second and a, ϱ, $\varkappa$, k, $\mathcal{L}_\mathrm{T}$, $\mathcal{L}_\varrho$ denote respectively the coefficient of thermal expansion, the density, the coefficient of thermometric conductivity, the wave number of the perturbation, the partial derivatives of $\mathcal{L}$ with respect to T, ϱ; both evaluated in the equilibrium state.

The effects of a uniform rotation and a uniform magnetic field on the thermal- convective instability of a stellar atmosphere have been studied, separately by Defouw [1] and by Bhatia [2]. The criterion for instability has been found to be the same, for situations of astrophysical interest.

However, the Hall effects which are very important for many cases of astrophysical interest, have not been included in the above studies. The object of the present paper is to include the rotation and Hall effects on the thermal- convective instability of a stellar atmosphere and to investigate the modification, if any, in the criterion for monotonic instability. Here we consider an infinite horizontal fluid layer acted on by a uniform vertical magnetic field $\boldsymbol{H}(0,0,H)$ and gravity force $\boldsymbol{g}(0,0,-g)$. This layer is heated from below such that a steady adverse temperature gradient $\beta\ (=\mathrm{d}T/\mathrm{d}z)$ is maintained. Also, the fluid layer is assumed to be rotating about the z-axis with uniform angular velocity $\boldsymbol{\Omega}(0,0,\Omega)$.

2. Perturbation Equations

Let δp, $\delta\varrho$, $\boldsymbol{q}(u,v,w)$ and $\boldsymbol{h}(h_x,h_y,h_z)$ denote the perturbations in pressure p, density ϱ, velocity and magnetic field $\boldsymbol{H}$, respectively; g, ν, η, N and e denote, respectively, the gravitational acceleration, the kinematic viscosity, the resistivity, the electron number density and charge of an electron. Then the linearized hydromagnetic perturbation equations appropriate to the problem are:

$$\varrho \frac{\partial \boldsymbol{q}}{\partial t} = -\nabla\,\delta p + \varrho\,\nu\,\nabla^2 \boldsymbol{q} + \frac{1}{4\pi}(\nabla\times\boldsymbol{h})\times\boldsymbol{H} + \boldsymbol{g}\,\delta\varrho + 2\,\varrho(\boldsymbol{q}\times\boldsymbol{\Omega})\,, \qquad (2)$$

$$\nabla\cdot\boldsymbol{q} = 0, \quad \nabla\cdot\boldsymbol{h} = 0\,, \qquad (3)$$

$$\frac{\partial \boldsymbol{h}}{\partial t} = (\boldsymbol{H}\cdot\nabla)\,\boldsymbol{q} + \eta\,\nabla^2\boldsymbol{h} - \frac{1}{4\pi N e}\nabla\times[(\nabla\times\boldsymbol{h})\times\boldsymbol{H}]\,. \qquad (4)$$

The first law of thermodynamics can be written as

$$C_\mathrm{v}\frac{\mathrm{d}T}{\mathrm{d}t} = -\mathcal{L} + \frac{K}{\varrho}\nabla^2 T + \frac{p}{\varrho^2}\frac{\mathrm{d}\varrho}{\mathrm{d}t}\,, \qquad (5)$$

where K, C_v, T and t denote respectively the thermal conductivity, the specific heat at constant volume, the temperature and the time.

Following Defouw [1], the linearized perturbation form of Eq. (5) is

$$\frac{\partial\Theta}{\partial t} + \frac{1}{C_\mathrm{p}}(\mathcal{L}_\mathrm{T} - a\,\varrho\,\mathcal{L}_\varrho)\,\Theta - \varkappa\nabla^2\Theta = -\left(\beta + \frac{g}{C_\mathrm{p}}\right) w\,. \qquad (6)$$

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where Θ is the perturbation in temperature. In obtaining (6), the Boussinesq equation of state $\delta\varrho = -\alpha\varrho\Theta$ has been used.

Here we consider the case in which both boundaries are free and the medium adjoining the fluid is non-conducting. The case of two free boundaries is the most appropriate for stellar atmospheres (Spiegel [3]).

The boundary conditions appropriate for the problem are (Chandrasekhar [4]):

$$w = \partial^2 w/\partial z^2 = 0\,,\ \Theta = 0\,,\quad \partial\zeta/\partial z = 0\,,$$

$\xi = 0$ and $\boldsymbol{h}$ is continuous with an external vacuum field.

Here ζ and ξ denote the z-components of vorticity and current density respectively.

3. Dispersion Relation and Discussion

Analyzing in terms of normal modes, we seek solutions whose dependence on x, y, z and t is of the form

$$\exp[i k_x x + i k_y y + n t] \sin k_z z\,, \tag{8}$$

where n is the growth rate and $k_z = s\pi/\mathrm{d}$, s being any integer and d is the thickness of the layer and $k(=\sqrt{k_x^2+k_y^2+k_z^2})$ is the wave number of the perturbation.

Eqs. (2) – (4) and (6) give

$$\frac{\partial}{\partial t}\nabla^2 w = g\alpha\left(\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}\right) + \nu\nabla^4 w + \frac{H}{4\pi\varrho}\nabla^2\frac{\partial}{\partial z}h_z - 2\Omega\frac{\partial\zeta}{\partial z}\,, \tag{9}$$

$$\frac{\partial\zeta}{\partial t} = \nu\nabla^2\zeta + \frac{H}{4\pi\varrho}\frac{\partial\xi}{\partial z} + 2\Omega\frac{\partial w}{\partial z}\,, \tag{10}$$

$$\frac{\partial h_z}{\partial t} = H\frac{\partial w}{\partial z} + \eta\nabla^2 h_z - \left(\frac{H}{4\pi N e}\right)\frac{\partial\xi}{\partial z}\,, \tag{11}$$

$$\frac{\partial\xi}{\partial t} = H\frac{\partial\zeta}{\partial z} + \eta\nabla^2\xi + \left(\frac{H}{4\pi N e}\right)\nabla^2\frac{\partial}{\partial z}h_z\,, \tag{12}$$

$$\frac{\partial\Theta}{\partial t} + D\Theta = -\left(\beta + \frac{g}{C_\mathrm{p}}\right) w\,. \tag{13}$$

Eliminating ζ, ξ, h_z and Θ from Eqs. (9) – (13) and using (8), we obtain the dispersion relation

$$n^7 + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0 = 0\,, \tag{14}$$

where

$$A_6 = D + 3k^2(\nu + 2\eta)\,,$$

$$A_5 = 3k^2 D(\nu+2\eta) + 2k^4(\nu^2 + 6\nu\eta + 5\eta^2) + \Gamma\left(\beta + \frac{g}{C_\mathrm{p}}\right) + 2k_z^2 V^2 + 4\Omega^2 k_z^2/k^2 + 2k_z^2 k^2\left(\frac{H}{4\pi N e}\right)^2,$$

$$\begin{aligned} A_4 = {} & 2\eta k^2 M + (D + \nu k^2)N' + k_z^2 V^2(D + \eta k^2) + 2k^2(\nu+2\eta)S + \{D + k^2(\nu+2\eta)\}R \\ & + \nu k^2 N' + \eta k^2 k_z^2 V^2 + \frac{4\Omega^2 k_z^2}{k^2}(D + 4\eta k^2)\,, \end{aligned}$$

$$\begin{aligned} A_3 = {} & 2k^2(\nu+2\eta)[2\eta k^2 M + (D+\nu k^2)N' + k_z^2 V^2(D+\eta k^2)] + \eta k^2 k_z^2 V^2 D \\ & + \{D + k^2(\nu+2\eta)\}\{\eta k^2 k_z^2 V^2 + \nu k^2 N'\} + M N' + R S \\ & + 4\Omega k_z^2\left[k_z^2 V^2\cdot\frac{H}{4\pi N e} + \Omega\left\{6\eta^2 k^2 + 2k_z^2\left(\frac{H}{4\pi N e}\right)^2 + 4\eta D\right\}\right], \end{aligned}$$

$$\begin{aligned} A_2 = {} & 2k^2(\nu+2\eta)\{M N' + \eta k^2 k_z^2 V^2\} + R[2\eta k^2 M + (D+\nu k^2)N' + k_z^2 V^2(D+\eta k^2)] \\ & + \left\{\eta k^2 k_z^2 V^2 + \nu\eta^2 k^6 + \nu k_z^2 k^4\left(\frac{H}{4\pi N e}\right)^2\right\} S + 16\Omega^2\eta k_z^2\left(N' + k_z^2 k^4 V^2\cdot\frac{H}{4\pi N e}\right) \\ & + \frac{k_z^2 D}{k^2}\left[8k^2\Omega^2\left\{3\eta^2 k^2 + k_z^2\left(\frac{H}{4\pi N e}\right)^2\right\} + 4\Omega k^2 k_z^2 V^2\cdot\frac{H}{4\pi N e}\right], \end{aligned} \tag{15}$$

$$A_1 = R[M N' + \eta k^2 k_z^2 V^2 D] + \{\eta k^2 k_z^2 V^2 + \nu k^2 N'\} \{2 \eta k^2 M + k_z^2 V^2 (D + \eta k^2) + (D + \nu k^2) N'\}$$
$$+ \frac{k_z^2}{k^2} \left[4 \Omega^2 N'^2 + k_z^4 k^4 V^4 \left(\frac{H}{4 \pi N e}\right)^2 + 4 \Omega k_z^2 k^2 V^2 \cdot \frac{H}{4 \pi N e} + 8 \Omega \eta k^2 D \left\{ 2 \Omega N' + k^2 k_z^2 V^2 \cdot \frac{H}{4 \pi N e} \right\} \right]$$
$$A_0 = (\eta k^2 k_z^2 V^2 / + \nu k^2 N') (M N' + \eta k^2 k_z^2 V^2 D) + \frac{k_z^2}{k^2} D \left[2 \Omega N' + k^2 k_z^2 V^2 \cdot \frac{H}{4 \pi N e} \right]^2,$$

where
$$\Gamma = \frac{g \alpha (k_x^2 + k_y^2)}{k^2}, \quad V^2 = \frac{H^2}{4 \pi \varrho},$$
$$M = \nu k^2 D + \Gamma \left(\beta + \frac{g}{C_p}\right), \quad N' = \eta^2 k^4 + \left(\frac{H}{4 \pi N e}\right)^2 k_z^2 k^2,$$
$$R = \eta^2 k^4 + 2 \nu \eta k^4 + k_z^2 V^2 + k_z^2 k^2 \left(\frac{H}{4 \pi N e}\right)^2,$$
and
$$S = k^2 D (\nu + 2 \eta) + \eta k^4 (\eta + 2 \nu) + \Gamma \left(\beta + \frac{g}{C_p}\right) + \left(\frac{H}{4 \pi N e}\right)^2 k_z^2 k^2 + k_z^2 V^2 .$$

In most cases of astrophysical interest, the effects of viscosity and resistivity are negligible. Setting $\nu = \eta = 0$, the dispersion relation (14) reduces to

$$\begin{aligned} n^7 + D n^6 &+ \left\{ \Gamma \left(\beta + \frac{g}{C_p}\right) + 2 k_z^2 V^2 + \frac{4 \Omega^2 k_z^2}{k^2} + 2 k_z^2 k^2 \left(\frac{H}{4 \pi N e}\right)^2 \right\} n^5 \\ &+ D \left\{ 2 k_z^2 V^2 + \frac{4 \Omega^2 k_z^2}{k^2} + 2 k_z^2 k^2 \left(\frac{H}{4 \pi N e}\right)^2 \right\} n^4 \\ &+ \Bigg[\left\{ k_z^2 V^2 + \left(\frac{H}{4 \pi N e}\right)^2 k^2 k_z^2 \right\}^2 + \Gamma \left(\beta + \frac{g}{C_p}\right) k_z^2 \left\{ V^2 + 2 k^2 \left(\frac{H}{4 \pi N e}\right)^2 \right\} \\ &\qquad + 4 \Omega k_z^4 \cdot \frac{H}{4 \pi N e} \left(V^2 + 2 \cdot \frac{H \Omega}{4 \pi N e} \right) \Bigg] n^3 \\ &+ D \left[\left\{ k_z^2 V^2 + k_z^2 k^2 \left(\frac{H}{4 \pi N e}\right)^2 \right\}^2 + 4 \Omega k_z^4 \cdot \frac{H}{4 \pi N e} \left(V^2 + 2 \frac{\Omega H}{4 \pi N e} \right) \right] n^2 \\ &+ k_z^4 k^2 \left(\frac{H}{4 \pi N e}\right)^2 \left[\Gamma \left(\beta + \frac{g}{C_p}\right) \left\{ V^2 + k^2 \left(\frac{H}{4 \pi N e}\right)^2 \right\} + 4 \Omega k_z^2 \cdot \frac{H}{4 \pi N e} \left(\frac{H \Omega}{4 \pi N e} + V^2 \right) + k_z^2 V^4 \right] n \\ &+ k_z^6 k^2 \left(\frac{H}{4 \pi N e}\right)^2 \left[V^2 + 2 \Omega \left(\frac{H}{4 \pi N e}\right) \right]^2 \cdot D = 0 . \end{aligned} \tag{16}$$

When the inequality (1) is satisfied i. e. $D < 0$, the constant term in (16) is negative. This means that Eq. (16) has one positive real root, meaning thereby monotonic instability.

We conclude, therefore, that the criterion for instability remains the same in the presence of rotation and Hall effects for the thermal-convective instability of a stellar atmosphere.

[1] R. J. Defouw, Astrophys. J. **160**, 659 [1970].
[2] P. K. Bhatia, Publ. Astr. Soc. Japan **23**, 181 [1971].
[3] E. A. Spiegel, Astrophys. J. **141**, 1068 [1965].
[4] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, p. 164, Clarendon Press, Oxford 1961.